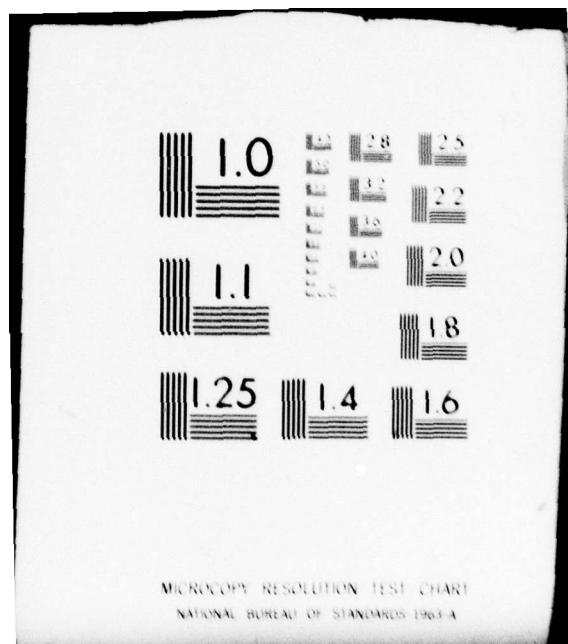


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6 CONVERGENCE RATES FOR THE MEAN INTEGRATED  
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DENSITY ESTIMATORS OF RECURSIVE  $\beta$ -FUNCTION TYPE.

by E. P. Cheng and R. J. Serfling

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## ABSTRACT

### CONVERGENCE RATES FOR THE MEAN INTEGRATED SQUARED ERRORS OF SOME NONPARAMETRIC DENSITY ESTIMATORS OF RECURSIVE $\delta$ -FUNCTION TYPE

For estimation of a probability density function  $f$  by an empirical function  $f_n$  based on a sample of size  $n$  from  $f$ , a widely used measure of goodness is the mean integrated squared error. For the well known  $\delta$ -function type of  $f_n$ , we show that the asymptotic behavior of this measure is essentially unchanged if  $f_n$  is replaced by a recursive version. Also, we characterize this asymptotic behavior under somewhat milder smoothness restrictions on  $f$  than previously considered in the literature, at the expense however of adding tail restrictions on  $f$ .



**Key phrases:** Nonparametric density estimation; mean integrated squared error; convergence rates; recursive estimators.

1. Introduction and a basic lemma. Consider estimation of an unknown probability density function  $f$  by an empirical function  $f_n$  based on a sample. A widely used measure of goodness is the mean integrated squared error. For the well-known  $\delta$ -function type of  $f_n$ , we show that the asymptotic behavior of this measure is essentially unchanged if  $f_n$  is replaced by a recursive version  $\tilde{f}_n$ . Also, we characterize the asymptotic behavior under somewhat milder conditions on  $f$  than previously considered in the literature, and compare with previous results.

Specifically, we consider "nonparametric" estimation of a density  $f$  ( $f(x) \geq 0$ ,  $\int f(x)dx = 1$ ) on the real line by an empirical function  $f_n$  (not necessarily itself a density) formed from a sample of independent random variables  $X_1, \dots, X_n$  each having density  $f$ . Various ways of concocting a useful  $f_n$  are reviewed in Wegman (1972), Fryer (1977), and Tapia and Thompson (1978). Here we consider the  $\delta$ -function (or in specialized form the *kernel* type) approach of Rosenblatt (1956), Whittle (1958), Parzen (1962), Leadbetter (1963), Watson and Leadbetter (1963), and Nadaraya (1974), in which  $f_n$  has the form

$$(1.1) \quad f_n(x) = n^{-1} \sum_{i=1}^n \delta_n(x - X_i), \quad -\infty < x < \infty,$$

where  $\delta_n(\cdot)$  is a suitably chosen approximant to the Dirac  $\delta$ -function.

The kernel type  $f_n$  corresponds to  $\delta_n$  of the form

$$\delta_n(u) = c_n^{-1} K(u/c_n), \quad -\infty < u < \infty,$$

where  $K(\cdot)$  is a specified "kernel" or weight function and  $\{c_n\}$  is a sequence of specified "bandwidth" constants tending to 0.

A useful global measure of goodness of an estimator  $f_n$  for  $f$  is the mean integrated squared error (MISE),

$$J(f, f_n) = E\left(\int [f_n(x) - f(x)]^2 dx\right).$$

Under restrictions on  $f$ , and for  $f_n$  chosen compatibly with respect to such restrictions, the rate of convergence of  $J(f, f_n)$  to 0 has been characterized by various authors. For  $\delta$ -function  $f_n$ , Watson and Leadbetter (1963) impose conditions on the characteristic function of  $f$ . For kernel type  $f_n$ , Nadaraya (1974) imposes conditions on the derivatives of  $f$  of order 2 and higher. The validity of such conditions may presuppose more knowledge of the unknown  $f$  than would be available realistically, thus making difficult the selection of a "compatible"  $f_n$ . In Section 2 we offer some competitive results under conditions on the tails of  $f$  (implied by simple moment conditions). Also, we extend Nadaraya's approach to the case of restrictions merely on the first derivative of  $f$ .

A variation on (1.1) is the associated recursive version defined by

$$(1.2) \quad \tilde{f}_n(x) = n^{-1} \sum_{i=1}^n \delta_i(x - X_i),$$

introduced in the context of kernel type  $f_n$  by Wolverton and Wagner (1969). We have  $\tilde{f}_n(x) = n^{-1}[(n-1)\tilde{f}_{n-1}(x) + \delta_n(x - X_n)]$ , so that the estimator need not be recomputed entirely when an additional observation is combined with previous ones. Besides saving computational effort, this lends itself to sequential sampling (see Davies and Wegman (1975)). On the other hand, there is the philosophical grievance that in (1.2), versus (1.1), the observations  $X_i$  are not being used symmetrically.

Nevertheless, for large samples, which is the case when the recursive approach is of greatest importance, this asymmetry does not affect efficiency, as we will see below. Also, (1.2) is structurally simpler than (1.1), in that (1.2) involves partial sums over the single sequence of random variables  $\{\delta_n(x - X_n), n \geq 1\}$ , whereas (1.1) entails the double array  $\{\delta_n(x - X_i), 1 \leq i \leq n, n \geq 1\}$ . Thus the application of classical probability theory is more straightforward and fruitful in connection with (1.2). For example, it is easy to state, pointwise in  $x$ , a law of the iterated logarithm for the estimator  $\tilde{f}_n(x)$ .

For kernel type  $f_n$ , the work of Parzen (1962) on pointwise mean square error of  $f_n(x)$  has been extended to recursive versions by Yamato (1971) and Wegman and Davies (1979). Winter (1978) has investigated a global measure, the uniform mean square error  $\sup_x E\{[\tilde{f}_n(x) - f(x)]^2\}$ . However, the MISE has not received attention. We now prove, for general  $\delta$ -function estimators, a simple lemma which yields the relation

$$(1.3) \quad \overline{\lim}_{n \rightarrow \infty} J(f, \tilde{f}_n) \leq \overline{\lim}_{n \rightarrow \infty} J(f, f_n),$$

showing that the recursive version  $\tilde{f}_n$  is asymptotically as effective as the nonrecursive counterpart. In proving the lemma, we will use the fact that by Fubini's theorem the MISE may be written as the "IMSE," which in turn may be written as the sum of the integrated variance and the integrated squared bias. That is, for any estimator  $f_n$ , we have

$$(1.4) \quad J(f, f_n) = A(f, f_n) + B(f, f_n),$$

where  $A(f, f_n) = \int E\{[f_n(x) - Ef_n(x)]^2\} dx$  and  $B(f, f_n) = \int [Ef_n(x) - f(x)]^2 dx$ .

LEMMA. For a given  $\delta$ -function sequence  $\{\delta_n\}$ , let  $f_n$  and  $\tilde{f}_n$  be given by (1.1) and (1.2), respectively. Then

$$(1.5) \quad J(f, \tilde{f}_n) \leq n^{-1} \sum_{i=1}^n J(f, f_i).$$

PROOF. We have

$$\begin{aligned} E\{[\tilde{f}_n(x) - E\tilde{f}_n(x)]^2\} &= \text{Var}\{n^{-1} \sum_{i=1}^n \delta_i(x - x_i)\} \\ &= n^{-2} \sum_{i=1}^n \text{Var}\{\delta_i(x - x_i)\} \\ &= n^{-2} \sum_{i=1}^n i \text{Var}\{f_i(x)\} \\ (1.6) \quad &\leq n^{-1} \sum_{i=1}^n \text{Var}\{f_i(x)\}. \end{aligned}$$

And, by Jensen's inequality,

$$\begin{aligned} [E\tilde{f}_n(x) - f(x)]^2 &= \{n^{-1} \sum_{i=1}^n [E\delta_i(x - x_i) - f(x)]\}^2 \\ &\leq n^{-1} \sum_{i=1}^n [E\delta_i(x - x_i) - f(x)]^2 \\ (1.7) \quad &= n^{-1} \sum_{i=1}^n [Ef_i(x) - f(x)]^2 \end{aligned}$$

By (1.6) and (1.7) we have  $A(f, \tilde{f}_n) \leq n^{-1} \sum_{i=1}^n A(f, f_i)$  and  $B(f, \tilde{f}_n) \leq n^{-1} \sum_{i=1}^n B(f, f_i)$ , from which (1.5) follows by 1.4).  $\square$

Thus  $J(f, \tilde{f}_n)$  is dominated by the Cesàro mean of  $\{J(f, f_i)\}$  and hence (1.3) follows.

2. On rates of convergence for the MISE. To establish a perspective, let us first review three key results in the literature. Watson and Leadbetter (1963) consider two forms of restriction on the rate of decrease

of the characteristic function  $\phi_f$  of the underlying density  $f$ . Either  $\phi_f$  has an exponential rate with coefficient  $\rho > 0$ , i.e.,

$$(2.1a) \quad |\phi_f(t)| \leq Ae^{-\rho|t|}, \text{ all } t, \text{ for some constant } A,$$

or it has an algebraic rate of degree  $p > 0$ , i.e.,

$$(2.2a) \quad \lim_{|t| \rightarrow \infty} |t|^p |\phi_f(t)| < \infty.$$

In this connection two types of  $\delta$ -sequence  $\{\delta_n\}$  for estimators given by (1.1) are considered. In each case  $\delta_n$  is assumed square integrable with Fourier transform  $\phi_{\delta_n}$ . Either  $\phi_{\delta_n}(t)$  has the form

$$(2.1b) \quad h(a_n e^{\alpha|t|}),$$

or it has the form

$$(2.2b) \quad h(a_n t),$$

where in either case  $h$  is a bounded square integrable function and  $\{a_n\}$  is a sequence of constants tending to 0. In the first case the auxiliary conditions

$$(2.1c) \quad |1 - h(t)| \leq B|t|, \quad |t| \leq 1, \text{ for some constant } B,$$

and

$$(2.1d) \quad a_n = Dn^{-b}, \text{ for } b > \frac{1}{2} \text{ and } b \geq \alpha/2\rho,$$

are required, and in the second case the auxiliary conditions

$$(2.2c) \quad \int |t|^{-2p} [1 - h(t)]^2 dt < \infty$$

and

$$(2.2d) \quad a_n = Dn^{-1/2p}.$$

In this setting Watson and Leadbetter (1963) establish: under conditions (2.1),

$$(A) \lim_{n \rightarrow \infty} (n/\log n) J(f, f_n) = Q_A;$$

under conditions (2.2),

$$(B) \lim_{n \rightarrow \infty} n^{1-1/2p} J(f, f_n) = Q_B.$$

Here  $Q_A$  and  $Q_B$  are specified finite constants whose values are not important in the present discussion.

We mention that (2.2b) equivalently means that  $\delta_n(u) = c_n^{-1} K(u/c_n)$ , with  $\theta_K(t) = h(t)$ , making  $f_n$  of kernel type with square integrable kernel  $K$ . In any case, in order to assert for an estimator  $f_n$  the rate (A) or (B) for  $J(f, f_n)$ , one must construct  $f_n$  so as to meet conditions (2.1b, c, d) or (2.2b, c, d). This requires prior knowledge of (2.1a) or (2.2a), respectively, for the unknown density  $f$ .

Alternatively, Nadaraya (1974) utilizes conditions on the derivatives of  $f$ . For even integer  $s \geq 2$ , let  $W_s$  denote the set of functions  $f(x)$  having derivatives of  $s$ -th order with  $f^{(s)}(x)$  being a bounded continuous  $L_2(-\infty, \infty)$  function, and let  $H_s$  denote the class of kernels  $K(u)$  satisfying  $K(u) = K(-u)$ ,  $\int K(u)du = 1$ ,  $\sup_u |K(u)| < \infty$ ,  $\int u^i K(u)du = 0$ ,  $1 \leq i \leq s-1$ ,  $\int u^s K(u)du = 0$ , and  $\int u^s |K(u)|du < \infty$ . For estimation of  $f$  satisfying

$$(2.3a) \quad f \in W_s,$$

the "compatible" kernel type estimator  $f_n$  corresponds to  $\delta_n(u) = c_n^{-1} K(u/c_n)$  with

$$(2.3b) \quad K \in H_s$$

and

$$(2.3c) \quad c_n = Cn^{-1/(2s+1)}.$$

Nadaraya (1974) establishes: under conditions (2.3),

$$(C) \quad \lim_{n \rightarrow \infty} n^{1-1/(2s+1)} J(f, f_n) = Q_C.$$

This result is competitive with (B), but for  $s \geq 2$  the condition (2.3a) presupposes considerable knowledge of the properties of  $f$ . We now establish two results requiring less stringent prior information. We restrict to kernel type  $f_n$ .

THEOREM 1. Let  $f$  have continuous square integrable derivative. Let  $K$  satisfy  $\int K(u)du = 1$  and  $\int u^2 |K(u)|du < \infty$ , and take  $c_n = Cn^{-1/3}$ . Then

$$(D) \quad J(f, f_n) = O(n^{-2/3}).$$

PROOF. Using (1.4), we treat the terms  $A(f, f_n)$  and  $B(f, f_n)$  separately. Assuming  $K$  square integrable and requiring nothing of  $f$ , we have

$$(2.4) \quad \begin{aligned} A(f, f_n) &= n^{-1} \int \text{Var}\{\delta_n(x - X_1)\}dx \leq n^{-1} \int E\{\delta_n^2(x - X_1)\}dx \\ &= (nc_n)^{-1} \int K^2(u)du. \end{aligned}$$

In dealing with the term  $B(f, f_n)$ , we apply Taylor's formula with integral form of remainder, to write

$$\begin{aligned} Ef_n(x) - f(x) &= \int K(u)[f(x - c_n u) - f(x)]du \\ &= c_n \int K(u) \frac{1}{0} \int f'(x - tc_n u)u dt du, \end{aligned}$$

giving

$$\begin{aligned} B(f, f_n) &\leq c_n^2 \int \left[ \int |K(u)| \int_0^1 |f'(x - tc_n u)u| dt du \right]^2 dx \\ &\leq c_n^2 \int \left[ \left( \int_0^1 |f'(x - tc_n u)| dt \right)^2 u^2 |K(u)| du \right] dx \\ (2.5) \quad &= c_n^2 \left( \int [f'(x)]^2 \left( \int u^2 |K(u)| du \right) \right). \end{aligned}$$

by two applications of Jensen's inequality. Combining (2.4) and (2.5) with  $c_n = Cn^{-1/3}$ , (D) follows.  $\square$

The preceding result extends the range of Nadaraya's conditions on  $f$  down to first order derivative restrictions. We have, in fact, used Nadaraya's line of argument in the above proof. In the next result we introduce a modification in the treatment of  $B(f, f_n)$  and further relax the conditions on smoothness of  $f$ , but at the expense of adding restrictions on the tails of  $f$ . For each real  $q > 0$ , we put

$$C_{f,q} = \sup_t t^q \left\{ \int_{|x|>t} f(x) dx \quad (\leq \int |x|^q f(x) dx) \right.$$

Also, put

$$C_{f,\infty} = \sup\{t : \int_{|x|\leq t} f(x) dx = 1\}.$$

We will use the following inequalities derived in Serfling (1979). For any probability density  $g$ ,

$$(2.6a) \quad \int |g(x) - f(x)| dx \leq 2C_{f,q}^{1/(q+1)} \sup_x |g(x) - f(x)|^{q/(q+1)}$$

for  $0 < q < \infty$ , and

$$(2.6b) \quad \int |g(x) - f(x)| dx \leq 2C_{f,\infty} \sup_x |g(x) - f(x)|.$$

We now attack again the term  $B(f, f_n)$ . Putting  $g_{n,u}(x) = f(x - c_n u)$  and assuming  $\int K(u)du = 1$ , we have

$$\begin{aligned} B(f, f_n) &= \iint \{[g_{n,u}(x) - f(x)]K(u)du\}^2 dx \\ &\leq \iint [g_{n,u}(x) - f(x)]^2 dx K(u)du. \end{aligned}$$

Since  $g_{n,u}(x)$ ,  $-\infty < x < \infty$ , is a probability density, we have by (2.6a) that

$$B(f, f_n) \leq 2C_{f,q}^{1/(q+1)} \int \sup_x |g_{n,u}(x) - f(x)|^{(2q+1)/(q+1)} du.$$

Now assuming that  $f$  is Lipschitz of order  $\beta$  on  $(-\infty, \infty)$ , we have

$$\sup_x |g_{n,u}(x) - f(x)| \leq A_f c_n^\beta |z|^\beta \text{ for some constant } A_f, \text{ and thus}$$

$$(2.7) \quad B(f, f_n) = O(c_n^{\beta(2q+1)/(q+1)}),$$

provided that  $C_{f,q} < \infty$  and  $\int |u|^{(2q+1)/(q+1)} |K(u)| du < \infty$ . Likewise, if  $C_{f,\infty} < \infty$ , we obtain  $O(c_n^{2\beta})$  in (2.7). Thus we have

**THEOREM 2.** Let  $f$  be Lipschitz of order  $\beta$  on  $(-\infty, \infty)$  and let  $C_{f,q} < \infty$ , where  $0 < q \leq \infty$ . Let  $K$  satisfy  $\int K(z)dz = 1$  and  $\int |z|^{\beta(2q+1)/(q+1)} |K(z)| dz < \infty$ . Take  $c_n = Cn^{-(q+1)/[(2\beta+1)q+1]}$ . Then

$$(E) \quad J(f, f_n) = O(n^{-\beta(2q+1)/[(2\beta+1)q+1]}).$$

In particular, for  $f$  Lipschitz of order 1 on  $(-\infty, \infty)$  and having bounded support,  $J(f, f_n) = O(n^{-2/3})$ , the same as in Theorem 1. For  $\beta < 1$  or  $q < \infty$ , however, the rate in (E) is slower than  $O(n^{-2/3})$ .

Conditions (A) - (E) comprise a hierarchy of possible rates for  $J(f, f_n)$ . Rates close to  $O(n^{-1})$  are achieved in (A), in (B) for  $p$  suffi-

ently large, and in (C) for  $s$  sufficiently large. On the other hand, the rates in (C) and (D) are no slower than  $O(n^{-4/5})$  and  $O(n^{-2/3})$ , respectively. The rates in (E) may be as high as  $O(n^{-2/3})$  but may be as slow as  $O(n^{-\gamma})$ , any  $\gamma > 0$ , as may be the rates in (B) for  $p$  sufficiently close to  $\frac{1}{2}$ .

Some interrelationships among conditions on the characteristic function of  $f$ , conditions on the derivatives of  $f$ , and conditions on the tails of  $f$  are as follows. If  $f$  has an integrable  $s$ -th derivative, then  $\Phi_f$  decreases faster than an algebraic rate  $p > s$  (Feller (1966), p. 487). On the other hand, it is readily proved that if  $\Phi_f$  decreases at algebraic rate of degree  $p > s$ ,  $s$  an integer, then  $f$  has a bounded continuous derivative of order  $s - 1$ . However, conditions on the tails of  $f$  relate to properties of  $\Phi_f$  at the origin. Namely, if  $f$  has  $q$ -th absolute moment,  $q$  an integer, then  $\Phi_f$  has a  $q$ -th derivative; conversely, if  $\Phi$  has a  $q$ -th order derivative at the origin, then  $f$  has all moments up to order  $q$  if  $q$  even,  $q - 1$  if  $q$  odd.

Finally, we remark that in view of the lemma of Section 1 the  $O(\cdot)$  rates specified in (A) - (E) apply equally well to  $J(f, \tilde{f}_n)$ , where  $\tilde{f}_n$  is the recursive counterpart of  $f_n$ .

#### REFERENCES

- [1] Davies, H. I. and Wegman, E. J. (1975). Sequential nonparametric density estimation. *IEEE Trans. Info. Th.* IT-21, 619-628.
- [2] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications*, Vol. II. Wiley, New York.
- [3] Fryer, M. J. (1977). A review of some non-parametric methods of density estimation. *J. Inst. Maths Applies.* 20, 335-354.

- [4] Leadbetter, M. R. (1963). On the Non-Parametric Estimation of Probability Densities. Research Triangle Institute Technical Report No. 11 (Doctoral dissertation, University of North Carolina).
- [5] Nadaraya, E. A. (1974). On the integral mean square error of some nonparametric estimates for the density function. *Theor. Prob. Prob. Applic.* 19, 133-141.
- [6] Parzen, E. (1962). On estimation of a probability density function and mode. *Ann. Math. Statist.* 33, 1065-1076.
- [7] Rosenblatt, M. (1956). Remarks on some nonparametric estimates of a density function. *Ann. Math. Statist.* 27, 832-837.
- [8] Serfling, R. J. (1979). A Variation on Scheffé's Theorem, with Application to Nonparametric Density Estimation. Florida State University Statistics Report N502 (ONR Technical Report No. 137).
- [9] Tapia, R. A. and Thompson, J. R. (1978). *Nonparametric Probability Density Estimation*. The Johns Hopkins University Press, Baltimore.
- [10] Watson, G. S. and Leadbetter, M. R. (1963). On the estimation of the probability density, I. *Ann. Math. Statist.* 34, 480-491.
- [11] Wegman, E. J. (1972). Nonparametric probability density estimation: I. A summary of available methods. *Technometrics* 14, 533-546.
- [12] Wegman, E. J. and Davies, H. I. (1979). Remarks on some recursive estimators of a probability density. *Ann. Statist.* 7, 316-327.
- [13] Whittle, P. (1958). On the smoothing of probability density functions. *JRSS B* 20, 334-343.
- [14] Winter B. B. (1978). Recursive Nonparametric Estimation of Densities and Some Density Functionals. Department of Mathematics, University of Ottawa.
- [15] Wolverton, C. T. and Wagner, T. J. (1969). Asymptotically optimal discriminant functions for pattern classification. *IEEE Trans. Info. Th.* IT-15, 258-265.
- [16] Yamato, H. (1971). Sequential estimation of a continuous probability density function and mode. *Bull. Math. Statist.* 14, 1-12.

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